VOLUME FUNCTIONS ON BLOW-UPS AND SESHADRI CONSTANTS

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Abstract. We exhibit some relations between the Seshadri constant of an ample divisor along a closed subscheme and the behaviour of the volume function on the corresponding blow-up. As an application, we give an equivalent formulation of Nagata's conjecture in terms of the differentiability of a real valued function.

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1. Introduction

Let X be a projective variety of dimension $d \geq 1$ over an algebraically closed field k. The volume of a Cartier divisor D on X measures the asymptotic growth of the linear systems $|nD|$, $n \in \mathbb{N}$:

$$\mathrm{vol}(D) = \limsup_{n \to \infty} \frac{h^0(X, nD)}{n^d/d!}.$$

This invariant depends only on the numerical equivalence class of D, and can be extended to define a continuous function

$$\mathrm{vol}\colon N^1(X)_{\mathbb{R}} \to \mathbb{R}$$

on the real Néron-Severi group $N^1(X)_{\mathbb{R}} = N^1(X) \otimes_{\mathbb{Z}} \mathbb{R}$ of X (see [12, section 2.2.C] and [5, section 2.4] for details). The big cone $\mathrm{Big}(X) \subseteq N^1(X)_{\mathbb{R}}$ is the convex cone of classes $\alpha \in N^1(X)_{\mathbb{R}}$ with $\mathrm{vol}(\alpha) > 0$. The study of the volume function is an interesting topic of research, as emphasized in the survey [9]. For example, it is believed that vol is real analytic on some "large" open subset of the big cone [9, Conjecture 2.18]. The differentiability of vol on $\mathrm{Big}(X)$ was established by Boucksom, Favre and Jonsson [3] in characteristic zero and by Cutkosky [5] in general.

In this paper we are interested in the behaviour of the volume function on blow-ups. More precisely, let D be an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X and let $Z \subsetneq X$ be a closed proper subscheme, with defining ideal I_Z. Let $\pi_Z\colon \widetilde{X} \to X$ be the blow up along I_Z and let $E = \pi_Z^{-1}(Z)$ be the exceptional divisor, with the scheme structure defined by the ideal $I_Z\mathcal{O}_{\widetilde{X}} \simeq \mathcal{O}_{\widetilde{X}}(1)$. For $t \in \mathbb{R}$, we let $D_t = \pi_Z^* D - tE$. The purpose of this short article is to observe that the behaviour of the function

$$\varphi_{D,Z}\colon \mathbb{R}_{\geq 0} \to \mathbb{R},\ t \mapsto \mathrm{vol}(D_t)$$

is closely related to the Seshadri constant of D along Z, defined to be

$$\epsilon_Z(D) = \sup\{t \in \mathbb{R} \mid D_t \text{ is ample}\}.$$

Seshadri constants were originally introduced by Demailly [7] in the case where $Z = \{x\}$ is a closed point. They have been widely studied since then, at least when $\dim Z = 0$, and have become an important tool to understand the geometry of projective varieties. We refer the reader to [2] and [12, Chapter 5] for background and motivation about this invariant. We shall prove the following theorem and study some of its consequences.

Theorem 1.1. *Assume that the line bundle $\mathcal{O}_E(-E) = \mathcal{O}_{\widetilde{X}}(1)_{|E}$ is nef on E. Then*

$$\epsilon_Z(D) = \sup\{t \geq 0 \mid \varphi_{D,Z} \textit{ is polynomial on } [0,t]\}.$$

We remark that the assumption on $\mathcal{O}_E(-E)$ is automatically satisfied when $Z = \{x_1, \ldots, x_r\}$ is a collection of $r \geq 1$ distinct closed points in X, since in that case the line bundle $\mathcal{O}_{\widetilde{X}}(1)$ is relatively ample. When the inclusion $Z \subsetneq X$ is a regular embedding (e.g. when X and Z are smooth), $\mathcal{O}_E(-E)$ is nef if and only if the anti-conormal bundle I_Z/I_Z^2 is nef. We shall actually prove a generalization of Theorem 1.1, valid without any positivity assumption on $\mathcal{O}_E(-E)$ (see Theorem 3.1). The proof of Theorem 1.1 builds on ideas of McKinnon and Roth [13], and is based on a criterion for amplitude in terms of higher cohomological functions due to de Fernex, Küronya and Lazarsfeld [6] and to Murayama [15] (see Theorem 2.1 below). We use the assumption on $\mathcal{O}_E(-E)$ to guarantee the vanishing of the higher cohomological functions $\widehat{h}^i(\widetilde{X}, D_t)$ when $i \geq 2$ and $t \geq 0$ (see Lemma 3.3), in order to combine the main result of [6] with the asymptotic Riemann-Roch theorem. This leads to the identities

$$\epsilon_Z(D) = \sup\{t \geq 0 \mid \mathrm{vol}(D_t) = D_t^{\dim X}\} = \sup\{t \geq 0 \mid \widehat{h}^1(\widetilde{X}, D_t) = 0\},$$

from which we easily deduce Theorem 1.1.

Let $\gamma_Z(D)$ be the supremum of the real numbers t such that $\varphi_{D,Z}(t) > 0$. We consider the function

$$\phi_{D,Z} \colon [0, \gamma_Z(D)] \to \mathbb{R}, \ t \mapsto \langle D_t^{\dim X - 1} \rangle \cdot E,$$

where $\langle D_t^{\dim X - 1} \rangle$ denotes the positive intersection product introduced in [3, section 2] (see also [5, section 4]). By [5, Theorem 5.6], the function $\varphi_{D,Z}$ is differentiable on $(0, \gamma_Z(D))$ and its derivative is given by $\varphi'_{D,Z} = -\dim(X)\phi_{D,Z}$. In particular, Theorem 1.1 remains true if $\varphi_{D,Z}$ is replaced by $\phi_{D,Z}$.

Corollary 1.2. *If $\mathcal{O}_E(-E)$ is nef, then*

$$\epsilon_Z(D) = \sup\{t \geq 0 \mid \phi_{D,Z} \textit{ is polynomial on } [0,t]\}.$$

When $\widetilde{X}$ is a smooth surface (for example when X and Z are smooth and $\dim X = 2$), $\varphi_{D,Z}$ is a piecewise polynomial function of degree at most 2 by a theorem of Bauer, Küronya and Szemberg [1, Theorem]. It follows that $\varphi_{D,Z}$ and $\phi_{D,Z}$ are polynomial on an interval $[0,t]$ if and only if $\phi_{D,Z}$ is differentiable on $(0,t)$.

Corollary 1.3. *Assume that $\widetilde{X}$ is a smooth surface and that $\mathcal{O}_E(-E)$ is nef. Then*

$$\epsilon_Z(D) = \sup\{t \geq 0 \mid \phi_{D,Z} \textit{ is differentiable on } (0,t)\}.$$

It is worth to note that when $\widetilde{X}$ is a smooth surface, the function $\phi_{D,Z}$ is simply given by $\phi_{D,Z}(t) = P_t \cdot E$ for all $t \in [0, \gamma_Z(D))$, where P_t is the positive part in the Zariski decomposition of $D_t = \pi_Z^* D - tE$. This follows from the definition of the positive intersection product, as explained in [3, section 3.4]. Let us also mention that when X is a smooth surface and $Z \subsetneq X$ is a smooth irreducible curve, Theorem 1.1 and Corollary 1.3 remain valid even if $\mathcal{O}_E(-E)$ is not nef (see Corollary 3.5).

Let us now focus on zero-dimensional subschemes, that is $Z = \{x_1, \ldots, x_r\}$ is a collection of closed points in X. When k is uncountable and the points $x_1, \ldots, x_r$

are in very general position, we shall see that the function $\varphi_{D,Z}$ depends only on D and r (see Proposition 4.2 for a precise statement, and Proposition 4.1 for a more general result valid for higher dimensional subschemes). We denote it by $\varphi_{D,r}$. It is differentiable on some non-empty interval $[0, \gamma_r(D))$, and we let $\phi_{D,r} = -\varphi'_{D,r}/\dim X$. In particular, we recover from Theorem 1.1 the well-known fact that Seshadri constants at r very general points all take the same value $\epsilon_r(D)$. Indeed, we have

$$\epsilon_r(D) = \sup\{t \geq 0 \mid \varphi_{D,r} \text{ is polynomial on } [0,t]\}.$$

Of course we also have analogues of Corollaries 1.2 and 1.3: for example, if X is a smooth surface then

$$\epsilon_r(D) = \sup\{t \geq 0 \mid \phi_{D,r} \text{ is differentiable on } (0,t)\}. \tag{1}$$

A celebrated conjecture of Nagata predicts that for $r \geq 9$ very general points $x_1, \ldots, x_r$ in $\mathbb{P}^2_{\mathbb{C}}$ and for any integral curve $C \subset \mathbb{P}^2_{\mathbb{C}}$, we have

$$\deg(C) \geq \frac{1}{\sqrt{r}} \sum_{i=1}^{r} \operatorname{mult}_{x_i} C.$$

This conjecture was settled by Nagata when r is a perfect square, but the general case remains open despite many attempts in the past decades. It can be formulated in terms of Seshadri constants as follows: if D is a line in $\mathbb{P}^2_{\mathbb{C}}$, then $\epsilon_r(D) \geq 1/\sqrt{r}$ whenever $r \geq 9$ ([12, Remark 5.1.14]). In view of (1), we have the following equivalent formulation of

Nagata's conjecture. *Let $X = \mathbb{P}^2_{\mathbb{C}}$ and D be a line. Then for any $r \geq 9$, the function $\phi_{D,r}$ is differentiable on $(0, 1/\sqrt{r})$.*

We mention that similar reformulations of Nagata's conjecture can also be derived from the computation of Newton–Okounkov bodies of line bundles on blow-ups of $\mathbb{P}^2_{\mathbb{C}}$ obtained by Eckl ([8, Theorems 3.4 and 3.5]).

Organization of the paper. We recall some preliminary results on higher cohomological functions in section 2. We then prove Theorem 1.1 in section 3. In section 4 we study the behaviour of $\varphi_{D,Z}$ when Z varies in a flat family of subschemes (Proposition 4.1). In the case $\dim Z = 0$, we prove that the function $\varphi_{D,Z} = \varphi_{D,r}$ depends only on D and $r = \operatorname{card}(Z)$ when Z consists of very general points (Proposition 4.2).

2. Conventions and background

Throughout this paper we work over an algebraically closed field k. We denote by $\operatorname{Div}(X)$ the group of Cartier divisors on a projective scheme X. A $\mathbb{R}$-Cartier $\mathbb{R}$-divisor on X is an element of $\operatorname{Div}(X)_{\mathbb{R}} := \operatorname{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. We let $N^1(X)_{\mathbb{R}}$ be the real Néron-Severi space of X ([12, section 1.3.B]). A projective variety is a reduced and irreducible projective scheme.

2.1. Higher cohomological functions.

Let X be a projective variety of dimension $d \geq 1$. For any integer $0 \leq i \leq d$, we denote by $\widehat{h}^i \colon \operatorname{Div}(X) \to \mathbb{R}$ the higher cohomological function introduced by Küronya in [11]: for any $D \in \operatorname{Div}(X)$ we have

$$\widehat{h}^i(X, D) = \limsup_{n \to +\infty} \frac{h^i(X, nD)}{n^d/d!},$$

where $h^i(X, nD) = h^i(X, \mathcal{O}_X(nD))$ is the dimension of $H^i(X, \mathcal{O}_X(nD))$ as a k-vector space. Note that $\widehat{h}^0(X, D) = \operatorname{vol}(D)$ coincides with the volume of D. Moreover, the functions $\widehat{h}^i$ are homogeneous of degree d and induce well defined functions

$$\widehat{h}^i \colon \operatorname{Div}(X)_{\mathbb{R}} \to \mathbb{R}$$

for $0 \leq i \leq d$, that are continuous on every finite-dimensional $\mathbb{R}$-linear subspace with respect to any norm (see [11, Corollary 5.3] and [4, Proposition 3.4.8]). We will use the following theorem characterizing ampleness in terms of vanishing of higher cohomological functions. It was proved by de Fernex, Küronya and Lazarsfeld [6, Theorem A] over the complex numbers, and generalized in arbitrary characteristic by Murayama [15].

Theorem 2.1 ([15], Theorem B). *Let $D, A \in \mathrm{Div}(X)_{\mathbb{R}}$ be two $\mathbb{R}$-Cartier $\mathbb{R}$-divisors on X, with A ample. Then D is ample if and only if there exists a real $\gamma > 0$ such that*

$$\widehat{h}^i(X, D - tA) = 0$$

for all $i \in \{1, \ldots, d\}$ and $t \in (0, \gamma)$.

2.2. **Cohomology of nef divisors.** We need the following technical lemma for the proof of Theorem 1.1.

Lemma 2.2. *Let $A_1, \ldots, A_\ell \in \mathrm{Div}(X)$ be nef Cartier divisors on a projective scheme X of dimension $d \geq 0$. Then for any coherent sheaf $\mathcal{F}$ on X, there exists a real number C such that*

$$h^i(X, \mathcal{O}_X(n_1 A_1 + \ldots + n_\ell A_\ell) \otimes_{\mathcal{O}_X} \mathcal{F}) \leq C \max_{1 \leq j \leq \ell} n_j^{d-i}$$

for any integers $i, n_1, \ldots, n_\ell \geq 0$.

Proof. We adapt the arguments of [12, Theorem 1.4.40]. We may assume by induction that the statement is true for any projective scheme of dimension $\leq d-1$. For any $\mathbf{n} = (n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$, we let $\mathbf{n}A \in \mathrm{Div}(X)$ be the nef Cartier divisor given by $\mathbf{n}A = \sum_{j=1}^{\ell} n_j A_j$. By Fujita's vanishing theorem [12, Theorem 1.4.35 and Remark 1.4.36], there exists a very ample divisor $H \in \mathrm{Div}(X)$ which does not contain any subvariety of X defined by the associated primes of $\mathcal{F}$ and such that

$$H^i(X, \mathcal{O}_X(H + \mathbf{n}A) \otimes_{\mathcal{O}_X} \mathcal{F}) = 0$$

for any $i \geq 1$ and $\mathbf{n} \in \mathbb{N}^\ell$. We have an exact sequence of sheaves

$$0 \to \mathcal{O}_X(\mathbf{n}A) \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{O}_X(H + \mathbf{n}A) \otimes_{\mathcal{O}_X} \mathcal{F} \to \mathcal{O}_H(H + \mathbf{n}A) \otimes_{\mathcal{O}_H} \mathcal{F}_{|H} \to 0,$$

from which we obtain

$$h^i(X, \mathcal{O}_X(\mathbf{n}A) \otimes_{\mathcal{O}_X} \mathcal{F}) \leq h^{i-1}(H, \mathcal{O}_H(H + \mathbf{n}A) \otimes_{\mathcal{O}_H} \mathcal{F}_{|H}) = O(\max_{1 \leq j \leq \ell} n_j^{d-i})$$

for any integer $i \geq 1$ by induction. Since the Euler characteristic

$$\chi(\mathcal{O}_X(\mathbf{n}A) \otimes_{\mathcal{O}_X} \mathcal{F}) = \sum_{i=0}^{d} (-1)^i h^i(X, \mathcal{O}_X(\mathbf{n}A) \otimes_{\mathcal{O}_X} \mathcal{F})$$

is a polynomial of total degree at most d in $n_1, \ldots, n_\ell$ [10, I.1], we deduce that $h^0(X, \mathcal{O}_X(\mathbf{n}A) \otimes_{\mathcal{O}_X} \mathcal{F}) = O(\max_{1 \leq j \leq \ell} n_j^d)$ and the lemma is proved. □

3. Proof of Theorem 1.1

Let X be a projective variety of dimension $d \geq 1$ and let $D \in \mathrm{Div}(X)_{\mathbb{R}}$ be an ample $\mathbb{R}$-Cartier $\mathbb{R}$-divisor. We fix a closed proper subscheme $Z \subsetneq X$, and we denote by $\pi_Z \colon \widetilde{X} \to X$ the blow-up of X along the ideal sheaf I_Z defining Z in X. We let E be the exceptional divisor and for any $t \in \mathbb{R}$, we consider the $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $D_t = \pi_Z^* D - tE$ on $\widetilde{X}$. Recall that the Seshadri constant of D along Z is

$$\epsilon_Z(D) = \sup\{t \in \mathbb{R}_{\geq 0} \mid D_t \text{ is ample}\}.$$

Since $\mathcal{O}_{\widetilde{X}}(-E) = \mathcal{O}_{\widetilde{X}}(1)$ is relatively ample with respect to π_Z, $\epsilon_Z(D) > 0$ is positive by [12, Proposition 1.7.10]. We also consider the invariant $\omega_Z(D) \in [0, \infty]$ defined by

$$\omega_Z(D) = \sup\{t \in \mathbb{R}_{\geq 0} \mid D_{t|E} \text{ is ample}\},$$

where by a slight abuse of notation we denoted by $D_{t|E}$ the image in $N^1(E)_\mathbb{R}$ of the class of D_t in $N^1(X)_\mathbb{R}$. Note that $\omega_Z(D) \geq \epsilon_Z(D)$. The goal of this section is to prove the following result, which generalizes Theorem 1.1 in the introduction.

Theorem 3.1. *We have*

$$\epsilon_Z(D) = \sup\{t \in [0, \omega_Z(D)) \mid \varphi_{D,Z} \textit{ is polynomial on } [0,t]\}.$$

Note that when $\mathcal{O}_E(-E)$ is nef, then $\omega_Z(D) = \infty$. Hence Theorem 3.1 indeed implies Theorem 1.1. We shall derive it from the following proposition.

Proposition 3.2. *We have*

$$\begin{aligned}\epsilon_Z(D) &= \sup\{t \in [0, \omega_Z(D)) \mid \operatorname{vol}(D_t) = D_t^d\} \\ &= \sup\{t \in [0, \omega_Z(D)) \mid \widehat{h}^1(\widetilde{X}, D_t) = 0\}.\end{aligned}$$

When $Z = \{x\}$ is a point, this statement is implicitly proved by McKinnon and Roth in [13, Proof of Theorem 9.1]. We shall prove Proposition 3.2 by combining the arguments of [13] with Lemma 2.2. We need the following lemma, which improves and generalizes [13, Lemma 4.1].

Lemma 3.3. *For any integer $i \in [2, d]$ and any real number $t \in [0, \omega_Z(D)]$, we have $\widehat{h}^i(\widetilde{X}, D_t) = 0$. Moreover,*

$$\operatorname{vol}(D_t) = D_t^d + \widehat{h}^1(\widetilde{X}, D_t).$$

Proof. We first assume that $D \in \operatorname{Div}(X)$ is a Cartier divisor. Let $i \in \{2, \ldots, d\}$ and $t \in [0, \omega_Z(D)]$. In order to prove that $\widehat{h}^i(\widetilde{X}, D_t) = 0$, we may assume that $t \in (0, \omega_Z(D)) \cap \mathbb{Q}$ by continuity of $\widehat{h}^i$. By homogeneity of $\widehat{h}^i$, we may even assume that t is a positive integer. For any integer $n \geq 1$, we have an exact sequence

$$0 \to \mathcal{O}_{\widetilde{X}}(nD_t) \to \mathcal{O}_{\widetilde{X}}(nD_0) \to \mathcal{O}_{\widetilde{X}}(nD_0) \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{ntE} \to 0, \tag{2}$$

where ntE denotes the subscheme of $\widetilde{X}$ defined by the nt-th power of the ideal sheaf I_E of the Cartier divisor E. Since D is ample, we have $H^j(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(nD_0)) = 0$ for all $n \gg 1$ and $j > 0$. It follows from (2) that

$$H^i(\widetilde{X}, nD_t) = H^{i-1}(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(nD_0) \otimes \mathcal{O}_{ntE}), \tag{3}$$

for n large enough. For any integer $\ell \geq 1$ we have $\mathcal{O}_E(-\ell E) = I_E^\ell / I_E^{\ell+1}$. This yields an exact sequence of sheaves on $\widetilde{X}$

$$0 \to \mathcal{O}_{\widetilde{X}}(nD_0) \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_E(-\ell E) \to \mathcal{O}_{\widetilde{X}}(nD_0) \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{(\ell+1)E} \to \mathcal{O}_{\widetilde{X}}(nD_0) \otimes_{\mathcal{O}_{\widetilde{X}}} \mathcal{O}_{\ell E} \to 0,$$

from which we obtain

$$h^{i-1}(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(nD_0) \otimes \mathcal{O}_{(\ell+1)E}) \leq h^{i-1}(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(nD_0) \otimes \mathcal{O}_{\ell E}) + h^{i-1}(E, \mathcal{O}_E(nD_0 - \ell E)).$$

Combining this inequality with (3), we get

$$h^i(\widetilde{X}, nD_t) = h^{i-1}(\widetilde{X}, \mathcal{O}_{\widetilde{X}}(nD_0) \otimes \mathcal{O}_{ntE}) \leq \sum_{\ell=0}^{nt-1} h^{i-1}(E, \mathcal{O}_E(nD_0 - \ell E)). \tag{4}$$

Let $0 \leq \ell \leq nt - 1$ be an integer and let $k = \lceil \ell / t \rceil$ be the least integer with $k \geq \ell / t$. We have

$$\mathcal{O}_E(nD_0 - \ell E) = \mathcal{O}_E((n-k)D_0 + kD_t + jE),$$

where $j = tk - \ell \in \{0, \cdots, t\}$. Since $\mathcal{O}_E(D_0)$ and $\mathcal{O}_E(D_t)$ are nef, Lemma 2.2 applied with $\mathcal{F} = \mathcal{O}_E(jE)$ for all possible values of j gives a real number $C > 0$ such that

$$h^{i-1}(E, \mathcal{O}_E(nD_0 - \ell E)) \leq Cn^{d-i}$$

for all $n \in \mathbb{N}$ and $\ell \in \{0, \dots, nt-1\}$. By (4), it follows that

$$\widehat{h}^i(\widetilde{X}, D_t) = \lim_{n \to +\infty} \frac{h^i(X, nD_t)}{n^d/d!} = 0.$$

By the asymptotic Riemann-Roch theorem, we have $\mathrm{vol}(D_t) - \widehat{h}^1(\widetilde{X}, D_t) = D_t^d$, and the lemma is proved in the case where $D \in \mathrm{Div}(X)$.

In general, there exist Cartier divisors $D_1, \dots, D_\ell \in \mathrm{Div}(X)$ such that $D \in V := \mathrm{Span}_{\mathbb{R}}(D_1, \dots, D_\ell)$, and there exists a sequence $(D_n)_{n\in\mathbb{N}}$ in $\mathrm{Span}_{\mathbb{Q}}(D_1, \dots, D_\ell) \subseteq V$ converging to D in V. Let $t \in (0, \omega_Z(D))$ be a real number. Then $D_{t|E}$ is ample, and therefore $D_{n,t|E}$ is ample for n sufficiently large by [12, Example 1.3.14]. In particular, $t \in (0, \omega_Z(D_n)]$ for $n \gg 1$. By the above and by homogeneity of the $\widehat{h}^i$, we have

$$\mathrm{vol}(D_{n,t}) - \widehat{h}^1(\widetilde{X}, D_{n,t}) = D_{n,t}^d \quad \text{and} \quad \widehat{h}^i(D_{n,t}) = 0 \ \forall\, i \in \{2, \dots, d\} \tag{5}$$

for $n \gg 1$. By continuity, (5) also holds for D and the lemma is proved. □

Proposition 3.2 is a consequence of Theorem 2.1 and Lemma 3.3.

Proof of Proposition 3.2. The second equality is implied by the first one by using Lemma 3.3. For any $t \in [0, \epsilon_Z(D)]$, D_t is nef and therefore $\mathrm{vol}(D_t) = D_t^d$ by [12, Corollary 1.4.41]. Since $t \leq \omega_Z(D)$, this gives the inequality

$$\epsilon_Z(D) \leq \sup\{t \in [0, \omega_Z(D)] \mid \mathrm{vol}(D_t) = D_t^d\}.$$

To prove the converse, let α be the supremum on the right hand side. By Lemma 3.3, we have $\widehat{h}^1(D_t) = 0$ for all $t \in [0, \alpha)$. Let $s \in (0, \epsilon_Z(D))$ and $t \in (0, \alpha)$ be real numbers. By definition of $\epsilon_Z(D)$, D_s is ample. For any $\lambda > 0$ small enough we have $0 \leq \frac{t-\lambda s}{1-\lambda} < \alpha$, and by homogeneity of $\widehat{h}^1$ we deduce that

$$\widehat{h}^1(\widetilde{X}, D_t - \lambda D_s) = (1-\lambda)^d \widehat{h}^1(\widetilde{X}, D_{\frac{t-\lambda s}{1-\lambda}}) = 0.$$

By homogeneity and Lemma 3.3, we also have $\widehat{h}^i(\widetilde{X}, D_t - \lambda D_s) = 0$ for all $i \geq 2$. By Theorem 2.1 D_t is ample, hence $t \leq \epsilon_Z(D)$. By letting t tend to α, we obtain

$$\sup\{t \in [0, \omega_Z(D)] \mid \mathrm{vol}(D_t) = D_t^d\} = \alpha \leq \epsilon_Z(D).$$

□

We can easily derive Theorem 3.1 from Proposition 3.2 as follows. Recall that $\varphi_{D,Z}(t) = \mathrm{vol}(D_t)$ for all $t \in \mathbb{R}_{\geq 0}$.

Proof of Theorem 3.1. By Proposition 3.2, it suffices to show that the following implication holds for any $\gamma > 0$:

$$\varphi_{D,Z} \text{ is polynomial on } [0,\gamma] \Longrightarrow \forall t \in [0,\gamma],\ \varphi_{D,Z}(t) = D_t^d.$$

Let $\gamma > 0$ be a real number such that $\varphi_{D,Z|[0,\gamma]}$ is a polynomial. Then

$$P \colon t \mapsto \varphi_{D,Z}(t) - D_t^d$$

is a polynomial function on $[0, \gamma]$. On the other hand, $P(t) = 0$ for all $t \in [0, \epsilon_Z(D)]$. Since $\epsilon_Z(D) > 0$, it follows that $P = 0$ on $[0, \gamma]$. □

In the end of this section, we study in more detail the case where $Z \subsetneq X$ is a smooth irreducible curve. The following example gives an alternative description of the invariant $\omega_Z(D)$.

Example 3.4. (1) Assume that X is a smooth surface and that $Z = C \subsetneq X$ is a smooth irreducible curve. In that case, $\widetilde{X} = X$ and $E = C$. In particular, $\mathcal{O}_E(-E)$ is nef if and only if $C^2 \leq 0$. On the other hand, $D_{t|E}$ is ample if and only if $D_t \cdot E = D \cdot C - tC^2 > 0$. It follows that $\omega_C(D) = \infty$ if $C^2 \leq 0$, and $\omega_C(D) = (D \cdot C)/C^2$ otherwise.

(2) Assume that X is smooth and that $Z = C \subsetneq X$ is a smooth irreducible curve. In that case, the normal sheaf $\mathcal{N}_C := (I_C/I_C^2)^\vee$ is a vector bundle of rank $d-1$ on C and we have $E = \mathrm{Proj}(\mathcal{N}_C^\vee)$. We denote by $\mu_{\max}(\mathcal{N}_C)$ the maximal slope of $\mathcal{N}_C$, defined as

$$\mu_{\max}(\mathcal{N}_C) = \max_{0 \neq \mathcal{F} \subseteq \mathcal{N}_C} \frac{\deg(\mathcal{F})}{\mathrm{rk}(\mathcal{F})},$$

where $\mathcal{F}$ runs over the non-zero sub-bundles of $\mathcal{N}_C$. We also define the strong maximal slope of $\mathcal{N}_C$ as $\overline{\mu}_{\max}(\mathcal{N}_C) = \sup_{\tau \colon C' \to C} \mu_{\max}(\tau^* \mathcal{N}_C)$, where the supremum is over the finite k-morphisms $\tau \colon C' \to C$ of smooth projective integral curves (when k has characteristic zero, we have $\overline{\mu}_{\max}(\mathcal{N}_C) = \mu_{\max}(\mathcal{N}_C)$ by [14, Proposition 7.1 (3)]). Let $\xi \in N^1(E)_{\mathbb{R}}$ be the class of $\mathcal{O}_E(1)$ and let $f \in N^1(E)_{\mathbb{R}}$ be the class of a fiber of $E \to C$. For any $t \in \mathbb{R}_{\geq 0}$, the class of $D_{t|E}$ in $N^1(E)_{\mathbb{R}}$ is $(D \cdot C)f + t\xi$ for any $t \geq 0$. By [14, Proposition 7.1 (3)], $D_{t|E}$ is ample if and only if $(D \cdot C) > t\overline{\mu}_{\max}(\mathcal{N}_C)$, and therefore

$$\omega_C(D) = \begin{cases} (D \cdot C)/\overline{\mu}_{\max}(\mathcal{N}_C) & \text{if } \overline{\mu}_{\max}(\mathcal{N}_C) > 0, \\ \infty & \text{if } \overline{\mu}_{\max}(\mathcal{N}_C) \leq 0. \end{cases}$$

We refer the reader to [16, section 3] for more explicit computations of the invariant $\omega_C(D)$ in the case where $X = \mathbb{P}^3_k$.

In view of Example 3.4 (1), we see that the assumption of Theorem 1.1 is rather restrictive when $\dim Z > 0$. Nevertheless, the following corollary shows that when $Z = C$ is a smooth irreducible curve in a smooth surface, Theorem 1.1 and Corollary 1.3 remain valid even if $\mathcal{O}_E(-E)$ is not nef. Recall that $\phi_{D,C}(t) = -\varphi'_{D,C}(t)/d$ for every $t \geq 0$ such that $\varphi_{D,C}(t) > 0$.

Corollary 3.5. *Assume that X is a smooth surface and that $Z = C \subsetneq X$ is a smooth irreducible curve. Then*

$$\begin{aligned} \epsilon_C(D) &= \sup\{t \geq 0 \mid \varphi_{D,C} \textit{ is polynomial on } [0,t]\} \\ &= \sup\{t \geq 0 \mid \phi_{D,C} \textit{ is differentiable on } (0,t)\}. \end{aligned}$$

Proof. If $C^2 \leq 0$, then $\mathcal{O}_E(-E)$ is nef and the result is given by Theorem 1.1 and Corollary 1.3. In the following, we assume that $C^2 > 0$. In particular, $\omega_C(D) = (D \cdot C)/C^2$ by Example 3.4 (1). Let

$$\beta_C(D) := \sup\{t \geq 0 \mid \varphi_{D,C} \text{ is polynomial on } [0,t]\},$$

and let $\gamma \in (0, \beta_C(D))$ be a real number. Then $\varphi_{D,C}$ is a decreasing polynomial function on $[0,\gamma]$. As in the proof of Theorem 3.1, we have

$$\varphi_{D,C}(t) = D_t^2 = D^2 - 2tD \cdot C + t^2 C^2$$

for every $t \in [0,\gamma]$. Since $\varphi_{D,C}(\gamma) = \mathrm{vol}(D_\gamma) \geq 0$, studying the variation of the polynomial $\varphi_{D,C|[0,\gamma]}$ leads to the inequalities

$$\gamma \leq \frac{D \cdot C - \sqrt{(D \cdot C)^2 - D^2 C^2}}{C^2} \leq \frac{D \cdot C}{C^2} = \omega_C(D)$$

(note that $(D \cdot C)^2 - D^2C^2 \geq 0$ by the Hodge index Theorem). Letting γ tend to $\beta_C(D)$, we obtain $\beta_C(D) \leq \omega_C(D)$. On the other hand we have $\epsilon_C(D) =$

$\min\{\beta_C(D), \omega_C(D)\}$ by Theorem 3.1, and therefore

$$\epsilon_C(D) = \beta_C(D) = \sup\{t \geq 0 \mid \varphi_{D,C} \text{ is polynomial on } [0,t]\}.$$

Combining this equality with [1, Theorem] as in the introduction, we also have

$$\epsilon_C(D) = \sup\{t \geq 0 \mid \phi_{D,C} \text{ is differentiable on } (0,t)\}.$$

□

Remark 3.6. Assume that X is smooth (of arbitrary dimension $d \geq 1$) and that $Z \subsetneq X$ is a smooth divisor with Picard rank $\rho(Z) = 1$. In that case, we also have

$$\epsilon_Z(D) = \beta_Z(D) := \sup\{t \geq 0 \mid \varphi_{D,Z} \text{ is polynomial on } [0,t]\}.$$

Indeed, if $t > \omega_Z(D)$ then $D_{t|Z}$ is not ample, hence it is not big since $\rho(Z) = 1$. Arguing as in the proof of Lemma 3.3, it follows that $\varphi_{D,Z}(t) = \varphi_{D,Z}(t')$ for all $t' > t$. This implies that $\omega_Z(D) \geq \beta_Z(D)$, and therefore $\epsilon_Z(D) = \beta_Z(D)$ by Theorem 3.1.

4. Variation of volume functions in families

We retain the notation of section 3, and we assume that the field k is uncountable. Our goal in this section is to study the behaviour of $\varphi_{D,Z}$ when the subscheme $Z \subsetneq X$ varies in a flat family. The following proposition implies in particular that the functions $\varphi_{D,Z}$ all coincide for sufficiently general zero-dimensional subschemes of a given cardinality, as claimed in the introduction (see Proposition 4.2 below).

Proposition 4.1. *Let S be a Noetherian scheme locally of finite type over k, and let $\mathcal{Z} \subsetneq X \times_k S$ be a closed subscheme, flat over S. For any $s \in S(k)$, let $\mathcal{Z}_s$ be the fiber of $\mathcal{Z}$ above s. Then there exists a countable union $V = \cup_{n\in\mathbb{N}} V_n \subsetneq S$ of proper subvarieties of S such that*

$$\varphi_{D,\mathcal{Z}_s}(t) = \varphi_{D,\mathcal{Z}_{s'}}(t)$$

for all $t \in \mathbb{R}_{\geq 0}$ and $s, s' \in S(k) \setminus V$.

Proof. We assume that S is irreducible without loss of generality. We first consider the case where $D \in \mathrm{Div}(X)$ is a Cartier divisor. Let $I_{\mathcal{Z}} \subset \mathcal{O}_{X\times_k S}$ be the ideal sheaf defining $\mathcal{Z}$ in $X \times_k S$, and let $s \in S(k)$. We identify the fiber $X \times_k \{s\}$ with X and we denote by $j_s \colon X \hookrightarrow X \times_k S$ the corresponding closed immersion. Let $\pi_s \colon \widetilde{X}_s \to X$ be the blow-up of X along the ideal $I_{\mathcal{Z}_s}$ defining $\mathcal{Z}_s$ in X, and let E_s be the exceptional divisor. By flatness, $j_s^* I_{\mathcal{Z}} = I_{\mathcal{Z}_s}$ is the ideal defining $\mathcal{Z}_s$ in X. By the semi-continuity theorem applied to $X \times_k S \to S$, the function

$$s \in S(k) \mapsto h^0(X, \mathcal{O}_X(pD) \otimes_{\mathcal{O}_X} I^q_{\mathcal{Z}_s})$$

is upper semicontinuous for any integers $p, q \geq 0$. On the other hand, we have $\pi_{s*}\mathcal{O}_{\widetilde{X}_s}(-qE) = I^q_{\mathcal{Z}_s}$ for any sufficiently large integer q, and therefore the projection formula gives

$$h^0(\widetilde{X}_s, p\pi_s^* D - qE_s) = h^0(X, \mathcal{O}_X(pD) \otimes I^q_{\mathcal{Z}_s})$$

for any integers $p, q \geq 0$ with q large enough. It follows that

$$\{s \in S(k) \mid \varphi_{D,\mathcal{Z}_s}(t) < \alpha\}$$

is a countable intersection of open subsets in $S(k)$ for any $t \in \mathbb{Q}_{\geq 0}$ and $\alpha \in \mathbb{R}$. This implies the existence of a countable union of proper subvarieties $V = \cup_{n\in\mathbb{N}} V_n \subsetneq S$ such that for all $s \in S(k) \setminus V$, we have

$$\forall\, t \in \mathbb{Q}_{\geq 0}, \quad \varphi_{D,\mathcal{Z}_s}(t) = \mathrm{vol}(\pi_s^* D - tE_s) = \inf_{s' \in S(k)} \varphi_{D,\mathcal{Z}_{s'}}(t). \tag{6}$$

By continuity of vol, (6) actually holds for all $t \in \mathbb{R}_{\geq 0}$. Therefore the function $\varphi_{D,\mathcal{Z}_s}$ is independent of the choice of $s \in S(k) \setminus V$, and the proposition is proved in the case where $D \in \mathrm{Div}(X)$ is a Cartier divisor.

In general, there exists a sequence $(D_n)_{n\in\mathbb{N}}$ of elements of $\mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ such that $(D_n)_{n\in\mathbb{N}}$ converges to D in $N^1(X)_{\mathbb{R}}$. By the above and by homogeneity of vol, for every $n \in \mathbb{N}$ there exists a countable union of proper subvarieties $W_n \subsetneq S$ such that $\varphi_{D_n,\mathcal{Z}_s}(t) = \varphi_{D_n,\mathcal{Z}_{s'}}(t)$ for all $t \in \mathbb{R}_{\geq 0}$ and all $s, s' \in S(k) \setminus W_n$. Let $V = \cup_{n\in\mathbb{N}} W_n$ and let $s, s' \in S(k) \setminus V$. Then for every $t \in \mathbb{R}_{\geq 0}$, we have

$$\varphi_{D,\mathcal{Z}_s}(t) = \lim_{n\to\infty} \varphi_{D_n,\mathcal{Z}_s}(t) = \lim_{n\to\infty} \varphi_{D_n,\mathcal{Z}_{s'}}(t) = \varphi_{D,\mathcal{Z}_{s'}}(t)$$

by continuity of vol. □

Proposition 4.1 applies in particular when S is a Hilbert scheme as follows. Given a very ample line bundle $\mathcal{O}_X(1)$ on X and a polynomial $\Psi \in \mathbb{Q}[T]$, we denote by $\mathrm{Hilb}^{\Psi}_{X/k}$ the Hilbert scheme of closed subschemes of X whose Hilbert polynomial computed with respect to $\mathcal{O}_X(1)$ is Ψ. By Proposition 4.1 applied to $S = \mathrm{Hilb}^{\Psi}_{X/k}$, there exist a countable union of proper subvarieties $V_\Psi \subsetneq \mathrm{Hilb}^{\Psi}_{X/k}$ and a function $\varphi_{D,\Psi} \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that

$$\varphi_{D,Z} = \varphi_{D,\Psi}$$

for every closed subscheme $Z \in \mathrm{Hilb}^{\Psi}_{X/k}(k) \setminus V_\Psi$. When we restrict our attention to zero-dimensional subschemes, we obtain the following result. Let $r \geq 1$ be an integer and let $X^r = X \times_k \cdots \times_k X$ be the fiber product of r copies of X.

Proposition 4.2. *There exist a countable union of proper subvarieties $V \subsetneq X^r$ and a function $\varphi_{D,r} \colon \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that $\varphi_{D,\{x_1,\dots,x_r\}} = \varphi_{D,r}$ for all $(x_1, \dots, x_r) \in X^r(k) \setminus V$.*

Proof. This follows from the above discussion by taking $\Psi = r$. Alternatively, this result follows directly from Proposition 4.1 applied to the schemes S and $\mathcal{Z} \subset X \times_k S$ defined by

$$S(k) = \{(x_1, \cdots, x_r) \in X^r_{\mathrm{sm}}(k) \mid x_i \neq x_j \ \forall i \neq j\}$$

and

$$\mathcal{Z}(k) = \{(x, (x_1, \dots, x_r)) \in X(k) \times S(k) \mid x \in \{x_1, \dots, x_r\}\},$$

where X_{sm} denotes the smooth locus of X. □

Remark 4.3. As mentioned in the introduction, Theorem 1.1 and Proposition 4.2 imply that the function

$$(x_1, \dots, x_r) \in X^r(k) \mapsto \epsilon_{\{x_1,\dots,x_r\}}(D) \in \mathbb{R}$$

is constant outside a countable union of proper subvarieties $V \subsetneq X^r$. This result is well-known and can also be derived directly from the definition of $\epsilon_{\{x_1,\dots,x_r\}}(D)$, by using that ampleness is an open condition in a proper family of line bundles (see [12, Theorem 1.2.17 and Example 5.1.11]).

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